The logical sustainability of the pension system

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Abstract. The sustainability of a pension system must be "logical" in type and not based on the forecasts optimism. This paper proposes a logical mathematical model to manage a pension system. An organic funded component of structural type is introduced. A very general differential-type condition is obtained under the assumption that the pension benefits are calculated according to the defined contribution scheme. A sufficient condition of easier application is also deduced. In the latter condition a control indicator is used which constitutes the natural change to the sustainability indicator, or Balance Ratio, of the Swedish pension system to make it logically based. This indicator can be decomposed into two components, one that manages the pay-as-you-go part of the pension system and the other that manages the funded part. The paper is developed in a non-steady-state situation, which has been quite neglected in the literature.

Keywords: pension system, pay-as-you-go system, funded system, social security.

1 Introduction

Pension system sustainability is an issue that is currently arousing great interest both at an operational level and a theoretical one.

Particularly for pure pay-as-you-go pension systems, financial sustainability in the middle and long term is based on the balanced ratio between the population of the pensioners and that of the contributors. Pure pay-as-you-go management works if the pension system can assure to the subscribers that future pension benefits are certain in a context of adequacy and fairness among the different generations. This occurs if the retirees to contributors ratio is sufficiently low and stable in time. On the contrary, as it has really happened over the last decades in different Western countries, if a demographic hole, and hence a contribution hole, is created to the shoulders of a generation, then the ratio between retirees and contributors is destined to grow significantly. Hence maintaining system management in the pure pay-as-you-go form does not guarantee in itself the adequacy of pension benefits in the perspective of inter-generational fairness.

This pension system analysis cannot therefore neglect considering the demographic structure of the underlying collectivity. Up to now the literature has paid great attention to the influence of economic and financial variables on pension systems, but has for the most part neglected demographic variables. Whenever the demographic issue has been considered, the underlying hypothesis was a steady state of population distribution by age, without considering the meaningful influences that fertility and mortality changes can have (Lee, 1994).

For pure pay-as-you-go pension systems the aim is to ensure the "logical certainty" for future benefits to pension system participants, in a perspective of adequacy and substantial fairness among generations.

In the author's opinion, the "logical certainty" derives precisely from the definition of logically sustainable pension system. A pension system is logically sustainable if it is characterized by the following three phases. The first can be called a "rules phase": this fixes the rules for contribution and for pension benefits calculation. The second or "control" phase sets well-founded logical and mathematical rules and indicators for controlling the financial sustainability of a pension system. The third or "rebalancing phase" establishes the modalities and the variables to be intervened on if the pension system has to be brought back to the sustainability levels provided for in the second phase.

In the perspective of logically sustainable pension schemes, in particular for pension systems that are not demographically stable in time, it is natural to introduce a structural type funded component (Angrisani, 2006).

This paper is structured in the following way. Section 2 proposes a logical mathematical model of a pension system founded on a mixed type financial management ("partial capitalization management"), which provides a structural funded component. This model can ensure the "logical certainty" of benefits since it uses a control indicator, namely the Logical Sustainability Indicator (LSI). The LSI is founded in logical mathematical sense for it is obtained using differential relationships, thereby ensuring pension system sustainability. The LSI can be reduced to two components, one which manages the pay-as-you-go part and the other which manages the funded part of the pension system. Furthermore, in Section 3, we show that the LSI is similar to the Balance Ratio, which is the real indicator used in the Swedish pension system as "a measure of the financial position of the system" (SSIA, 2007).

The Swedish indicator is based on Turnover Duration, which is "...equal to the time difference between the average age of retirees ... and the average age of contributors ... Both ages are money-weighted..." (Settergren, 2005, p. 121). This definition is obtained by means of an equation that "...informs the conceivably intuitively reasonable fact that in steady state, the liability divided by contributions is equal to the time difference between the average age of retirees ... and the average age of contributors..." (Settergren, 2005, p. 121). Unlike the Balance Ratio, the LSI proposed in this paper has a "logical-mathematical effectiveness". Section 4 subsequently provides conditions useful to pension system "stabilization".

2 Model description

The pension system is a defined contribution system. In order to simplify the exposition but without loss of generality, disability pension and survivor benefits are not included in the pension calculation. The pension is calculated dividing the total pension credit by an annuity divisor at the time of retirement. Divisors take into account a pre-paid interest rate or *technical rate*, which is equal to zero, and then coincide with the remaining life expectancy, since survivor pensions are not paid out. The pension is revalued annually by the rate returned to the whole system pension debt, rate net of the implicit demographic rate.

A deterministic framework is considered. All the following functions introduced are defined, on a yearly basis, at time t, for each t belonging to the time interval $[t^*, +\infty)$, with t^* as the initial time. It is, therefore, supposed that these are continuous and, when necessary, differentiable in the time interval $[t^*, +\infty)$.

For each t belonging to the time interval $[t^*, +\infty)$, the following functions are defined.

$F(t)$ is the pension system fund, that is the aggregate value of assets.
$r(t)$ is the instantaneous rate (i.e. intensity) of return on assets.
$C(t)$ is the instantaneous flow of pension contributions.
$P(t)$ is the instantaneous flow of pension disbursements.

Furthermore, let $L^T(t)$ denote the total pension liability. It comprises the pension liability both to contributors and to retirees, that is

$$L^T(t) = L^A(t) + L^P(t),$$

where $L^A(t)$ is the pension liability to contributors at the time t, defined as the *latent pension liability*, and $L^P(t)$ is the pension liability to retirees at time t, defined as the *current pension liability.*

The current pension liability, which to simplify the notation we have indicated by $L^P(t)$, depends also on the life-table survival function, $l(t)$, available at time t, that is $L^P(t) = L^P(t, l(t))$. This dependence on $l(t)$ will be highlight only if it is necessary.

Always referring to the time t,

${}^AL^P(t)$ indicates the instantaneous flow of pension liability which turns from latent into current;
$r_L^A(t)$ indicates the instantaneous rate of return on pension liability to contributors; and
$r_L^P(t)$ indicates the instantaneous rate of return on pension liability to retirees.

In the following let us assume that $\boldsymbol{r_L^A(t) = r_L^P(t) = r_L(t)}$, where $r_L(t)$ is the instantaneous rate of return on total pension liability (both current and latent).

In this scheme, we restrict ourselves to considering the downward effects of mortality only for the current component of pension liability. These effects are therefore implicitly included in $r_L^P(t)$, because in a defined contribution pension scheme the effects of a progressive extension of life expectancy have influence only on retirees pension liability.

Definition 1 Let $l(t)$ be the life-table survival function calculated at time t. Function $r_L^{\mathrm{ID}}(t)$ is the instantaneous rate of implicit demographic return (ID rate) and is defined by

$$r_L^{\mathrm{ID}}(t) = \lim_{\Delta t \to 0^-} \frac{L^P(t, l(t+\Delta t)) - L^P(t, l(t))}{\Delta t} \cdot \frac{1}{L^P(t, l(t))}. \tag{1}$$

The ID rate depends, not only on biological parameters, but also on the demographic structure of the retiree group. In the logic of the considered scheme of defined contribution type, the instantaneous rate of the implicit demographic return $r_L^{\mathrm{ID}}(t)$ is included in the instantaneous rate of return on pension liability to retirees, $r_L^P(t)$, which we have assumed as being equal to $r_L(t)$. Then the rate of return, which has to **explicitly** credit to current pension liability by the pension system, will be equal to $r_L^P(t) - r_L^{\mathrm{ID}}(t)$, so that the total rate really returned to the pension liability to retirees is equal to $r_L^P(t)$.

Let us consider the evolution in asset dynamics that arises from interest rate at time t, from contributions (which we consider as being already free of pension system management costs) and from pension payments. The assets evolution equation is

$$\dot{F}(t) = F(t)r(t) + C(t) - P(t). \tag{2}$$

Let us write the pension liability evolution equations respectively for contributors and retirees as:

$$\begin{aligned} \dot{L}^A(t) &= L^A(t)r_L(t) + C(t) - {}^AL^P(t) \\ \dot{L}^P(t) &= L^P(t)r_L(t) + {}^AL^P(t) - P(t). \end{aligned}$$

As regards total pension liability, we have the following evolution equation

$$\dot{L}^T(t) = L^T(t)r_L(t) + C(t) - P(t). \tag{3}$$

The latter formula (3) indicates the evolution of total pension system liability for a defined contribution pension scheme based on an actuarial equilibrium between contributions and pensions.

It must be pointed out that, in our assumptions, *the whole contribution amount* turns into pension liability, that is into pension benefits for subscribers, also including, for example, the contributions of those who have died in the course of their working years.

Definition 2 The divisor of total pension liability in the current pension liability the quotient of total pension liability and current pension liability at time t is indicated by $\nu(t)$, i.e.

$$\nu(t) = \frac{L^T(t)}{L^P(t)}, \quad \text{with } \nu(t) \geq 1.$$

Hence $L^P(t) = \dfrac{L^T(t)}{\nu(t)}$. It follows that $\dfrac{1}{\nu(t)}$ is the transformation coefficient of the total pension liability in the current pension liability.

Definition 3 The divisor of current pension liability in pension disbursements at time t is indicated by $\gamma(t)$, i.e.

$$\gamma(t) = \frac{L^P(t)}{P(t)}.$$

It is easily verified that $\gamma(t)$ is equal to the weighted mean of residual life expectancies for retirees at time t. In this mean the relative weight of each pensioner equals that of his pension in relation to the total weight of current pensions.

Definition 4 The divisor of total pension liability in pension disbursements is given by $\gamma(t)\nu(t)$. It satisfies the following

$$P(t) = \frac{L^T(t)}{\gamma(t)\nu(t)}. \tag{4}$$

This product transforms the total pension liability in a pension disbursement flow.

Furthermore, for each time t belonging to $[t^*, +\infty)$, the following real valued functions are defined, where

$W(t)$ is the instantaneous flow of wages and **can assume only positive values** and

$\alpha(t)$ is the contribution rate, with $\alpha(t) \geq 0$.

As it results that $C(t) = \alpha(t)W(t)$, and using formula (4), the total pension liability dynamics can be also expressed as

$$\begin{aligned}\dot{L}^T(t) &= L^T(t)r_L(t) + \alpha(t)W(t) - \frac{L^T(t)}{\gamma(t)\nu(t)} \\ &= L^T(t)\left(r_L(t) - \frac{1}{\gamma(t)\nu(t)}\right) + \alpha(t)W(t). \end{aligned} \tag{5}$$

Definition 5 The unfunded pension liability is indicated by $L^{\mathrm{UN}}(t)$ and is given by

$$L^{\mathrm{UN}}(t) = L^T(t) - F(t) \tag{6}$$

with, in general, $F(t) \leq L^T(t)$.

It is easy to verify that the expression of $L^{\mathrm{UN}}(t)$ time derivative can be obtained by means of formulae (2) and (3), and is given by

$$\dot{L}^{\mathrm{UN}}(t) = L^T(t)r_L(t) - F(t)r(t). \tag{7}$$

DEFINITION 6 The level of unfunded pension liability in relation to wages is indicated by $\beta(t)$ and is given by

$$\beta(t) = \frac{L^{\mathrm{UN}}(t)}{W(t)}. \tag{8}$$

DEFINITION 7 The level of unfunded contribution rate is indicated by $\alpha^{\mathrm{UN}}(t)$ and is given by

$$\alpha^{\mathrm{UN}}(t) = \frac{\beta(t)}{\gamma(t)\nu(t)}. \tag{9}$$

This is the level of contribution rate necessary for "covering" the unfunded pension disbursements.

DEFINITION 8 The difference between the real contribution rate and the level of unfunded contribution rate is defined as the level of funded contribution rate and is given by

$$\alpha^F(t) = \alpha(t) - \alpha^{\mathrm{UN}}(t) = \alpha(t) - \frac{\beta(t)}{\gamma(t)\nu(t)}. \tag{10}$$

The level of funded contribution rate **can also assume negative values**. By means of the formula (10), the real contribution rate can be split into two parts, the level of unfunded contribution rate and the level of funded contribution rate, that is

$$\alpha(t) = \alpha^{\mathrm{UN}}(t) + \alpha^F(t).$$

The following theorem will also use the following definition:

DEFINITION 9 The intrinsic instantaneous rate of return is indicated by $\mathrm{intr}\,(t)$ and is defined as

$$\mathrm{intr}\,(t) = r(t) - \frac{1}{\gamma(t)\nu(t)}. \tag{11}$$

THEOREM 1 (THE NECESSARY AND SUFFICIENT CONDITION FOR PENSION SYSTEM SUSTAINABILITY) *Let a pension system have an initial fund $F(t^*)$ greater than or equal to 0, i.e. $F(t^*) = F^* \geq 0$.*

The pension system is sustainable in the time interval $[t^*, t_f]$[1]*, if and only if for each time $t \in [t^*, t_f]$ the whole of the funded contribution, paid until the time t and discounted at time t^* by the intrinsic instantaneous rate of return, intr (t), does not create a deficit greater than the initial available fund $F(t^*)$, i.e.*

$$\textit{for each time } t \in [t^*, t_f] \quad F(t) \geq 0$$

if and only if

$$\textit{for each time } t \in [t^*, t_f] \textit{ it results}$$

[1]Note that the instant time t_f can also be equal to $+\infty$.

$$-\int_{t^*}^{t} e^{-\int_{t^*}^{\tau}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}\right)\mathrm{d}s} W(\tau)\,\alpha^F(\tau)\,\mathrm{d}\tau \leq F(t^*). \tag{12}$$

Proof. Let us reformulate (2) expressing contributions by means of the wage and contribution rate, namely $C(t) = \alpha(t)W(t)$, and pensions by means of the formula (4)

$$\dot{F}(t) = F(t)r(t) + C(t) - P(t) = F(t)r(t) + \left[\alpha(t)W(t) - (\gamma(t)\nu(t))^{-1}L^T(t)\right].$$

Adding and subtracting the same quantity $(\gamma(t)\nu(t))^{-1}F(t)$ at the second member, we obtain

$$\begin{aligned}\dot{F}(t) &= F(t)r(t) + \left[\alpha(t)W(t) - (\gamma(t)\nu(t))^{-1}(L^T(t) - F(t))\right] - (\gamma(t)\nu(t))^{-1}F(t)\\ &= F(t)r(t) + \left[\alpha(t)W(t) - (\gamma(t)\nu(t))^{-1}L^{\mathrm{UN}}(t)\right] - (\gamma(t)\nu(t))^{-1}F(t).\end{aligned}$$

In the latter, by means of (8), we obtain the following evolution equation of assets $F(t)$:

$$\dot{F}(t) = F(t)[r(t) - (\gamma(t)\nu(t))^{-1}] + W(t)\left(\alpha(t) - \frac{\beta(t)}{\gamma(t)\nu(t)}\right), \tag{13}$$

where $r(t) - (\gamma(t)\nu(t))^{-1}$ is the intrinsic instantaneous rate of return, indicated by intr (t) in (11), and $\alpha(t) - \dfrac{\beta(t)}{\gamma(t)\nu(t)}$ is the level of funded contribution rate defined in (10). It should be noted that $\beta(t)$ is the level of unfunded pension liability in relation to wages expressed by (8).

Taking into account the initial condition $F(t^*) = F^* \geq 0$, the evolution equation of assets $F(t)$, expressed by (13), involves the following explicit relationship between the assets $F(t)$ and the level of funded contribution rate, provided by (10), $\alpha^F(t) = \alpha(t) - \dfrac{\beta(t)}{\gamma(t)\nu(t)}$:

$$\begin{aligned}F(t) = e^{\int_{t*}^{t}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}\right)\mathrm{d}s}\Bigg(&F(t^*)\\ &+\int_{t^*}^{t} e^{-\int_{t^*}^{\tau}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}\right)\mathrm{d}s} W(\tau)\left(\alpha(\tau) - \frac{\beta(\tau)}{\gamma(\tau)\nu(\tau)}\right)\mathrm{d}\tau\Bigg).^{2}\end{aligned} \tag{14}$$

The latter formula (14) underlines well the "additive effect" of the level of funded contribution rate, $\alpha^F(t)$, on the evolution of the assets $F(t)$. In fact, if

[2] In (14) we can consider that the $\beta(t)$ values are directly and independently determined by the evolution equations of $F(t)$ and $L^T(t)$, respectively equations (2) and (3).

$\alpha^F(t)=0$, that is $\alpha(t)=\dfrac{\beta(t)}{\gamma(t)\nu(t)}=\alpha^{\mathrm{UN}}(t)$, then the "additive effect" is equal to zero.

By means of (14) we have that for each time $t\in[t^*,\,t_f]$ $F(t)\geq 0$ if and only if for each time $t\in[t^*,\,t_f]$

$$\left(F(t^*)+\int_{t^*}^{t} e^{-\int_{t^*}^{\tau}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}\right)\mathrm{d}s} W(\tau)\left(\alpha(\tau)-\frac{\beta(\tau)}{\gamma(\tau)\nu(\tau)}\right)\mathrm{d}\tau\right)\geq 0,$$

and therefore if and only if (12) holds. □

3 Relationship with the Balance Ratio of the Swedish pension system

Proposition 1 (Sufficient condition for pension system sustainability) *Assuming that $F(t^*)\geq 0$, sufficient condition for pension system sustainability in a given time interval $[t^*,\,t_f]$ is that for each time $t\in[t^*,\,t_f]$ the contribution rate $\alpha(t)$ is greater than or equal to the level of unfunded contribution rate, i.e.*

If

$$\text{for each time } t\in[t^*,\,t_f]\quad \alpha(t)\geq\alpha^{\mathrm{UN}}(t)=\frac{\beta(t)}{\gamma(t)\nu(t)} \tag{15}$$

then

for each time $t\in[t^*,\,t_f]$ $F(t)\geq 0$.

Proof. This derives directly from the previous Theorem 1. In fact, if (15) is true, then the condition (12) of Theorem 1 is satisfied, and so the pension system is sustainable. □

The sustainability condition, used in (15), allows us to define a sustainability indicator which proves to be similar to the sustainability indicator, called the Balance Ratio, used in the Swedish pension system. Unlike the latter, the sustainability indicator, which will be obtained through (15), is based on the sufficient condition of Proposition 1.

Let us then consider the condition (15). By means of formulae (8) and (6), this is equivalent to

$$\alpha(t)\geq\frac{1}{\gamma(t)\nu(t)}\frac{L^T(t)-F(t)}{W(t)}\quad\text{for each time } t\in[t^*,\,t_f],$$

which can be expressed as

$$\frac{\alpha(t)W(t)\gamma(t)\nu(t)+F(t)}{L^T(t)}\geq 1\quad\text{for each time } t\in[t^*,\,t_f]$$

with $L^T(t) > 0$. Then, if $F(t^*) \geq 0$, the sufficient condition for pension system sustainability in the time interval $[t^*, t_f]$ is equivalent to

$$\frac{C(t)\gamma(t)\nu(t) + F(t)}{L^T(t)} \geq 1 \quad \text{for each time } t \in [t^*, t_f]. \tag{16}$$

The following should be considered.

Definition 10 The Logical Sustainability Indicator of a pension system is indicated by LSI (t) and is given by

$$\text{LSI}\,(t) = \frac{C(t)\gamma(t)\nu(t) + F(t)}{L^T(t)}.$$

As mentioned above, condition (16) is equivalent to condition (15). Therefore the sufficient condition for pension system sustainability in the time interval $[t^*, t_f]$, assuming that $F(t^*) \geq 0$, is that

$$\text{LSI}\,(t) \geq 1 \quad \text{for each time } t \in [t^*, t_f]. \tag{17}$$

It should be noted that the LSI has an analytical form similar to that of the Balance Ratio indicator, which is used in the Swedish pension system as "a measure of the financial position of the system" (SSIA, 2007). The divisor of total pension liability in pension disbursements, $\gamma(t)\nu(t)$, can be considered the logical substitute of the Turnover Duration used in the Balance Ratio definition. Unlike the Turnover Duration, the divisor of total pension liability in pension disbursements is always defined and does not require the steady state hypothesis. Differently from the Balance Ratio, the LSI is logically based because it is founded on the sufficient condition of Proposition 1.

In the following we use the additional definitions.

Definition 11 The degree of PAYG covering of the pension disbursements is indicated by $D_c^{\text{PAYG}}(t)$ and is given by

$$D_c^{\text{PAYG}}(t) = \frac{C(t)}{P(t)}. \tag{18}$$

This indicates the pension disbursements level "covered" by the contributions.

Definition 12 The degree of funding of pension liability is indicated by $D_c(t)$ and is given by

$$D_c(t) = \frac{F(t)}{L^T(t)}. \tag{19}$$

This indicates the pension liability level "covered" by the assets.

The sustainability condition (16) can also be expressed as

$$\frac{\frac{\alpha(t)W(t)}{L^T(t)}}{\gamma(t)\nu(t)} + \frac{F(t)}{L^T(t)} \geq 1 \qquad \text{for each time } t \in [t^*,\, t_f],$$

and, taking into account (4), it follows that

$$\frac{C(t)}{P(t)} + \frac{F(t)}{L^T(t)} \geq 1 \qquad \text{for each time } t \in [t^*,\, t_f].$$

Therefore, by means of formulae (18) and (19), the sustainability condition in the time interval $[t^*,\, t_f]$, assuming that $F(t^*) \geq 0$, can be written as

$$D_c^{\text{PAYG}}(t) + D_c(t) \geq 1 \qquad \text{for each time } t \in [t^*,\, t_f].$$

The latter relationship provides the LSI indicator meaning. The two methods which are useful for managing a pension system, namely the PAYG and the funded methods, must work together to ensure sustainability. If the two types of pension system management produce an aggregate effect greater than 1, then the pension system is logically sustainable.

4 Further relationships

Further relationships useful for studying the sustainability of a defined contribution pension system can be proved.

PROPOSITION 2 *Let us assume that* $0 \leq F(t^*) < L^T(t^*)$.

For each time $t \in [t^*,\, t_f]$ $\quad \dot{\beta}(t) = 0$, *and hence* $\beta(t) = \beta(t^*)$,

if and only if

$$\textit{for each time } t \in [t^*,\, t_f] \quad r_L(t) = r(t)\frac{F(t)}{L^T(t)} + \frac{\dot{W}(t)}{W(t)}\frac{L^T(t) - F(t)}{L^T(t)}. \tag{20}$$

Proof. For each time $t \in [t^*,\, t_f]$, calculating the time derivative of the level of unfunded pension liability in relation to wages, $\beta(t)$ defined by formula (8), it follows that

$$\dot{\beta}(t) = \frac{\dot{L}^{\text{UN}}(t)W(t) - L^{\text{UN}}(t)\dot{W}(t)}{W^2(t)}.$$

By substituting in the previous formula the expressions of $L^{\text{UN}}(t)$ and of its time derivative, respectively (6) and (7), we obtain

$$\dot{\beta}(t) = \frac{(L^T(t)r_L(t) - F(t)r(t))W(t) - (L^T(t) - F(t))\dot{W}(t)}{W^2(t)}.$$

Then

$$\dot{\beta}(t) = 0$$

if and only if

$$r_L(t) = r(t)\frac{F(t)}{L^T(t)} + \frac{\dot{W}(t)}{W(t)}\frac{L^T(t) - F(t)}{L^T(t)}. \qquad \square$$

Remark 1 Using the definition expressed by (19), in condition (20) $r_L(t)$ can be expressed also as

$$r_L(t) = r(t)D_c(t) + \frac{\dot{W}(t)}{W(t)}(1 - D_c(t)).$$

Remark 2 We observe that in general

$$\frac{\dot{\beta}(t)}{\beta(t)} = \frac{\dot{L}^{\mathrm{UN}}(t)}{L^{\mathrm{UN}}(t)} - \frac{\dot{W}(t)}{W(t)} \tag{21}$$

and then the following holds too

$$\text{for each time } t \in [t^*,\, t_f] \qquad \dot{\beta}(t) = 0$$

if and only if

$$\text{for each time } t \in [t^*,\, t_f] \qquad \frac{\dot{L}^{\mathrm{UN}}(t)}{L^{\mathrm{UN}}(t)} = \frac{\dot{W}(t)}{W(t)}.$$

Proposition 3 *Let us assume that* $0 < F(t^*) < L^T(t^*)$.

For each time $t \in [t^*,\, t_f]$, $\dot{D}_c(t) = 0$, *and hence* $D_c(t) = D_c(t^*)$,

if and only if

$$\text{for each time } t \in [t^*,\, t_f] \qquad \alpha(t) = \frac{L^T(t)}{W(t)}\left[\frac{1}{\gamma(t)\nu(t)} - \frac{F(t)(r(t) - r_L(t))}{\beta(t)W(t)}\right]. \tag{22}$$

Proof. Let t be an instant of time belonging to the time interval $[t^*,\, t_f]$. Let us calculate the derivative in relation to time of the degree of funding of pension liability. By means of equations (2) and (3), this derivative can be expressed as in the following

$$\begin{aligned}\dot{D}_c(t) &= \frac{1}{(L^T(t))^2}\big[(F(t)r(t) + C(t) - P(t))L^T(t) \\ &\quad - F(t)(L^T(t)r_L(t) + C(t) - P(t))\big] \\ &= \frac{1}{(L^T(t))^2}\big[F(t)L^T(t)(r(t) - r_L(t)) + (C(t) - P(t))(L^T(t) - F(t))\big]\end{aligned}$$

and hence

$$\dot{D}_c(t) = 0$$

if and only if

$$C(t) - P(t) = -\frac{F(t)L^T(t)(r(t) - r_L(t))}{L^T(t) - F(t)}. \tag{23}$$

Through algebraic calculation, expressing contributions $C(t)$ by means of wage and contribution rate, namely $C(t) = \alpha(t)W(t)$, and pensions $P(t)$ by means of the formula (4), and using (8), we can obtain (22). □

Remark 3 Let us assume that the necessary and sufficient condition (22) holds. Substituting (23) into the time derivative of $F(t)$, expressed in (2), it follows that

$$\dot{F}(t) = F(t)r(t) - \frac{L^T(t)F(t)(r(t) - r_L(t))}{L^T(t) - F(t)},$$

and then

$$\dot{F}(t) = \frac{F(t)}{L^T(t) - F(t)}(L^T(t)r_L(t) - F(t)r(t)).$$

Dividing, therefore, both sides of the previous equation by $F(t)$, taking into account the time derivative of $L^{\mathrm{UN}}(t)$, see formula (7), and the expression (6), it follows that

$$\text{for each time } t \in [t^*, t_f] \quad \dot{D}_c(t) = 0$$

if and only if

$$\text{for each time } t \in [t^*, t_f] \quad \frac{\dot{F}(t)}{F(t)} = \frac{\dot{L}^{\mathrm{UN}}(t)}{L^{\mathrm{UN}}(t)}. \tag{24}$$

By means of (21) in Remark 2, for each time $t \in [t^*, t_f]$ it follows that $\dot{D}_c(t) = 0$ if and only if

$$\text{for each time } t \in [t^*, t_f] \quad \frac{\dot{F}(t)}{F(t)} = \frac{\dot{\beta}(t)}{\beta(t)} + \frac{\dot{W}(t)}{W(t)}. \tag{25}$$

Proposition 4 *Let us assume that* $0 < F(t^*) < L^T(t^*)$. *For each time* $t \in [t^*, t_f]$, *let us assume:*

$$r_L(t) = r(t)\frac{F(t)}{L^T(t)} + \frac{\dot{W}(t)}{W(t)}\frac{L^T(t) - F(t)}{L^T(t)} \qquad \textit{(hypothesis A)}$$

$$\alpha(t) = \frac{L^T(t^*)}{W(t^*)}\frac{1}{\gamma(t)\nu(t)} - \frac{F(t^*)}{W(t^*)}\left(r(t) - \frac{\dot{W}(t)}{W(t)}\right) \qquad \textit{(hypothesis B)}$$

then for each time $t \in [t^*, t_f]$ *we obtain* $\beta(t) = \beta(t^*)$ *and* $D_c(t) = D_c(t^*)$.

Proof. Let us assume that hypothesis A is true. By means of Proposition 2, for each time $t \in [t^*, t_f]$ it follows that $\dot{\beta}(t) = 0$ if and only if hypothesis A holds. Then for each time $t \in [t^*, t_f]$ we have $\beta(t) = \beta(t^*)$.

Let us consider the difference $r(t) - r_L(t)$. By replacing in the latter difference the expression of $r_L(t)$ derived from hypothesis A, it follows that

$$\begin{aligned} r(t) - r_L(t) &= r(t)\left(\frac{L^T(t) - F(t)}{L^T(t)}\right) - \frac{\dot{W}(t)}{W(t)}\left(\frac{L^T(t) - F(t)}{L^T(t)}\right) \\ &= \left(\frac{L^T(t) - F(t)}{L^T(t)}\right)\left(r(t) - \frac{\dot{W}(t)}{W(t)}\right). \end{aligned}$$

Multiplying both sides of the previous equality by $\dfrac{F(t)}{L^T(t) - F(t)}$, taking into account formula (6), i.e. $L^{\mathrm{UN}}(t) = L^T(t) - F(t)$, we obtain

$$\frac{F(t)}{L^{\mathrm{UN}}(t)}(r(t) - r_L(t)) = \frac{F(t)}{L^T(t)}\left(r(t) - \frac{\dot{W}(t)}{W(t)}\right). \tag{26}$$

Proposition 3 establishes that for each time $t \in [t^*, t_f]$ we obtain $\dot{D}_c(t) = 0$ if and only if (22), that is

$$\alpha(t) = \frac{L^T(t)}{W(t)}\left[\frac{1}{\gamma(t)\nu(t)} - \frac{F(t)}{L^{\mathrm{UN}}(t)}(r(t) - r_L(t))\right].$$

By (26), we obtain

$$\begin{aligned} \alpha(t) &= \frac{L^T(t)}{W(t)}\left[\frac{1}{\gamma(t)\nu(t)} - \frac{F(t)}{L^T(t)}\left(r(t) - \frac{\dot{W}(t)}{W(t)}\right)\right] \\ &= \frac{L^T(t)}{W(t)}\frac{1}{\gamma(t)\nu(t)} - \frac{F(t)}{W(t)}\left(r(t) - \frac{\dot{W}(t)}{W(t)}\right). \end{aligned} \tag{27}$$

From hypothesis A it follows that $\dot{\beta}(t) = 0$ for each time $t \in [t^*, t_f]$, and by means of Remark 3, it follows that for each time $t \in [t^*, t_f]$

$$\frac{\dot{F}(t)}{F(t)} = \frac{\dot{W}(t)}{W(t)},$$

and by means of Remark 2 it also follows that for each time $t \in [t^*, t_f]$

$$\frac{\dot{L}^{\mathrm{UN}}(t)}{L^{\mathrm{UN}}(t)} = \frac{\dot{W}(t)}{W(t)}.$$

Integrating the latter two relationships between t^* and t, with $t \in [t^*, t_f]$, we obtain respectively that for each time $t \in [t^*, t_f]$

$$\begin{aligned} \frac{F(t)}{W(t)} &= \frac{F(t^*)}{W(t^*)} \\ \frac{L^{\mathrm{UN}}(t)}{W(t)} &= \frac{L^{\mathrm{UN}}(t^*)}{W(t^*)}. \end{aligned} \tag{28}$$

Therefore it follows that for each time $t \in [t^*, t_f]$

$$\frac{L^T(t)}{W(t)} = \frac{L^T(t^*)}{W(t^*)}. \tag{29}$$

Consequently, by substituting the expressions (28) and (29) into the expression (27) of contribution rate $\alpha(t)$, the following is obtained:

$$\alpha(t) = \frac{L^T(t^*)}{W(t^*)}\frac{1}{\gamma(t)\nu(t)} - \frac{F(t^*)}{W(t^*)}\left(r(t) - \frac{\dot{W}(t)}{W(t)}\right). \tag{30}$$

□

5 Conclusions

In this paper we have proposed a new logical and mathematical model to manage a defined contribution pension system with a structural funded component. Within such a context a very general condition for the logical sustainability of a pension system is given. This is obtained by the model evolution being driven by differential equations. Furthermore a sufficient condition for easier application is provided. This is founded on a logical sustainability control indicator, defined as the LSI, in which other two sustainability indicators work: one manages the pay-as-you-go component of the pension system and the other manages the funded component. This LSI is compared to the Swedish Balance Ratio. Conditions useful for pension system "stabilization" are also provided.

It should be noted that the steady state hypothesis is not used in this paper.

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