A NECESSARY SUSTAINABILITY CONDITION FOR PARTIALLY FUNDED PENSION SYSTEMS

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ABSTRACT

This paper presents a necessary condition on the sustainability for defined contribution pension systems with a funded component under the assumption of a constant contribution rate and in the stabilisation phase.

The work finds classification in the framework of "logically sustainable pension systems", introduced in Angrisani (2006; 2008), which are pension systems whose sustainability is founded on logical-mathematical rules rather than on actuarial forecasting.

On the base of this theory and by assuming an efficient rule on the rate of return on the pension liability, we give the condition relative to the "minimum level" of the constant contribution rate, only necessary for the sustainability of pension systems in the stabilisation phase. The paper also gives the numerical example where the necessary condition of sustainability is applied.

Keywords: Public Pension Systems; Defined Contribution Pension System; Pension System Sustainability.

1. INTRODUCTION

In almost all Continental European countries the public pension systems are constructed on the pay-as-you-go (PAYGO) principle. The social, economic and demographic changes have strongly affected the long-term sustainability of these pension schemes. In particular, in the last decades the demographic trends highlight a progressive increase of life expectancy together with a contemporaneous reduction in fertility rates. In Italy, e.g., numerically reduced generations followed the baby-boom generations of the sixties. So, a "demographic wave" is expected in the Italian pension system beginning from the next thirties. In many European industrialized countries the demographic forecasting detects a considerable increase of the old-age dependency ratio, which will double on average in the next forty decades.

As a result, the demographic pressure will make public PAYGO pension systems unsustainable in a not too far future. In some European countries pension reforms have been already actuated, as in Italy, Sweden, Latvia and Poland. They have adopted Notional Defined Contribution (NDC) pension systems. There is a large and well-known literature which describes economic and actuarial features of NDC systems, see, e.g., Holzmann and Palmer 2006. In particular, the Swedish pension system is one of the most analyzed and debated in literature, see, e.g., Palmer 2000.

The Swedish pension system is a NDC type, with PAYGO management supported by a funded component, called Buffer Fund, whose function is to absorb "… interperiod discrepancies between pension contributions and pension expenditure. The primary purpose of the buffer fund is to stabilize pension disbursements and/or pension contributions in relation to economic and demographic variations…", SSIA 2009, p. 75. Furthermore, the Swedish pension system is provided with an automatic balance mechanism, which aims at restoring long-run balance in the system according to the Swedish sustainability indicator, the Balance Ratio. For a complete and exhaustive description of the Swedish pension system see, e.g., SSIA 2009.

Hence, in consideration of the necessity of pension reforms, which are founded on logical and mathematical rules rather than on actuarial projections, this paper analyses the sustainability issue for defined contribution pension systems with a funded component applying the methodology introduced in two previous works of Angrisani 2006, 2008. Such a methodology realizes a dynamic integration between the two methods of financial management, the PAYGO and the funded plan, respectively, which "… are not to be seen antithetical, as it commonly happens, but working as "the two ends of the same stick", in a "continuum"…", Angrisani 2006. The new methodological approach consists in considering the total pension liability subdivided in two components, one covered by the fund, and hence funded, and the other equal to its complimentary, and hence unfunded. The second paper, Angrisani 2008, develops the mathematical framework in which the variables of state and control for the pension system sustainability are defined. Furthermore, it gives a logic and mathematical condition of sustainability, which works under very general assumptions and not in the steady state.

In this paper we apply the cited methodology. In particular, we use the necessary and sufficient condition (NSC) of sustainability, introduced in Angrisani 2008, only as the necessary condition, and the specific rule on the efficient and sustainable rate of return on the pension liability, produced in Angrisani 2006. We work under the assumption of a constant contribution rate and in the stabilisation phase. We give the condition relative to the "minimum level" of the constant contribution rate only necessary for the pension system sustainability, in presence of a funded component. We also give numerical illustration of the necessary condition of sustainability.

This paper is structured as follows. In Section 2, the model of the pension scheme, which uses the new methodology of financial management, is reviewed, for an exhaustive and detailed description see Angrisani 2008. Furthermore, we briefly review the indicators and the general conditions of sustainability of these pension systems, already introduced in Angrisani 2006, 2008. In Section 3, the contribution rate is assumed to be constant over time. In this case, using the above cited NSC of sustainability (only as necessary condition) and assuming an efficient hypothesis on the rate of return on the pension liability, formulated in Angrisani 2006, 2008, the necessary condition of sustainability is given for these pension schemes in the stabilisation phase. Furthermore, the numerical illustration of the sustainability necessary condition is given. Our main conclusions are presented in Section 4.

2. THE MODEL

A defined contribution pension system provided with a funded component is taken into consideration. Pension benefits are calculated dividing the total pension credit by an annuity divisor at the time of retirement. The annuity divisor depends on the technical rate, namely a pre-paid rate of interest, which is chosen equal to zero. Disability pension and survivor benefits are not taken into account in the pension calculation.

We consider a fixed time interval *T*, that is $T = \left[t_*, t_f\right]$ with t_* and t_f defined as the initial and final instant of time, respectively; t_f can also be equal to $+\infty$. All the following functions are defined over *T* and have the necessary regularity for the next formalisations. The functions of instantaneous flows are evaluated on a yearly basis.

For each *t* in *T*, we have that:

$\alpha(t)$ is the contribution rate, with $\alpha(t) \geq 0$

$C(t)$ and $W(t)$ are the instantaneous flow of contributions and the instantaneous flow of wages, respectively, with $C(t) \geq 0$, $W(t) > 0$, and $C(t) = \alpha(t)W(t)$

$P(t)$ is the instantaneous flow of pension expenditure at time *t*, with $P(t) > 0$

$F(t)$ is the pension system assets, also called as pension system fund

$r(t)$ is the instantaneous rate of return on assets

$L^A(t)$ is the pension liability to contributors (the *latent pension liability*), with $L^A(t) \geq 0$

$L^P(t)$ is the pension liability to retirees (the *current pension liability*), with $L^P(t) > 0$

$L^T(t)$ is the total pension liability, with $L^T(t) > 0$ and $L^T(t) = L^A(t) + L^P(t)$.

The following definition is given.

Definition 1. *A pension system is sustainable in time interval T if and only if* $F(t) \geq 0$ *for each t in T.*

Function $F(t)$ is connected to contributions $C(t)$ and pension expenditure $P(t)$ by the basic differential equation

$$\dot{F}(t) = F(t)r(t) + C(t) - P(t). \qquad (1)$$

The change in the pension system assets is equal to the return on the assets plus the difference between contributions and pension expenditure.

Another basic equation of evolution is given for the total pension liability of the pension system. It is assumed that the rate of return on the latent and current components of the pension liability is the same. Hence, let $r_L(t)$ be the instantaneous rate of return on the total pension liability. One has to take into account that retirees benefits earn an implicit return deriving by the progressive extension of life expectancy; relating to this point, see the definition of the "implicit demographic rate" in Angrisani 2008, p. 70. Therefore, the rate of interest that has to be explicitly returned to the pension liability for retirees is equal to the difference between rate $r_L(t)$ and the implicit demographic rate.

Furthermore, we highlight that the whole contribution is assumed to turn into the pension liability, including, for example, the contributions of those who died during their working life. The differential equation for evolution of the total pension liability is given by

$$\dot{L}^{T}(t) = L^{T}(t) r_L(t) + C(t) - P(t). \qquad (2)$$

This equation uses two important control variables of the pension system, which are the rate of return on the pension liability, $r_L(t)$, and the contribution rate, $\alpha(t)$.

Definition 2. *For each instant t in time interval T, the unfunded pension liability is* $L^{UN}(t) = L^{T}(t) - F(t)$.

It is assumed that $L^{T}(t) \geq F(t)$ for all values *t* in *T*. The unfunded pension liability is then subjected to the condition $L^{UN}(t) \geq 0$, for all values *t* in *T*.

The following two indicators of the pension system state are also used. Function $D_c^{PAYGO}(t)$ measures the proportion of the current pension expenditure which is covered by current contributions. It is called the *degree of PAYGO covering of the pension expenditure by contributions* and, for each *t* in *T*, it is defined by

$$D_c^{PAYGO}(t) = \frac{C(t)}{P(t)}.$$

Function $D_c(t)$ indicates the *degree of funding of the pension liability; for each t in T it is given by*

$$D_c(t) = \frac{F(t)}{L^{T}(t)}$$

with condition $0 \leq D_c(t) \leq 1$. A degree of funding of the pension liability equal to 1 means that the system is fully funded, whereas a degree of funding equal to 0 means that the system is zero funded. It also results in $1 - D_c(t) = \frac{L^{UN}(t)}{L^{T}(t)}$, for each *t* in *T*.

Indicators of the model

According to definitions already introduced in Angrisani 2008, we review the following indicators of the state of the pension system, each of which is defined for each *t* in *T*.

Function $\nu(t)$ is the divisor of the total pension liability in the current pension liability at time t, i.e., $\nu(t)=\frac{L^T(t)}{L^P(t)}$, with $\nu(t)\geq 1$.

Function $\gamma(t)$ is the divisor of the current pension liability in the pension expenditure at time t, i.e., $\gamma(t)=\frac{L^P(t)}{P(t)}$.

Function $\gamma(t)\nu(t)$ is the divisor of the total pension liability in the pension expenditure at time t, i.e., $\gamma(t)\nu(t)=\frac{L^T(t)}{P(t)}$.

It has to be emphasised that the principal feature of these pension schemes consists in having the pension liability split up into two components, the unfunded component and the funded component, which are managed in a dynamic and integrated way. So, the introduction of $\gamma(t)\nu(t)$ naturally allows one to also consider the unfunded and the funded component relative to the pension expenditure, as it results in

$$P(t)=\frac{L^T(t)}{\gamma(t)\nu(t)}=\frac{L^{UN}(t)}{\gamma(t)\nu(t)}+\frac{F(t)}{\gamma(t)\nu(t)}.$$

The "unfunded pension expenditure", expressed by the first term, $\frac{L^{UN}(t)}{\gamma(t)\nu(t)}$, is the pension expenditure relative to the payment of the pension liability not covered by the fund. The "funded pension expenditure" is expressed by the second term, $\frac{F(t)}{\gamma(t)\nu(t)}$.

Function $\beta(t)$ is the level of the unfunded pension liability respect to wages at time t, i.e.,

$$\beta(t)=\frac{L^{UN}(t)}{W(t)}. \qquad (3)$$

Function $\alpha^{UN}(t)$ is the level of the unfunded contribution rate at time t, i.e.,

$$\alpha^{UN}(t) = \frac{\beta(t)}{\gamma(t)\nu(t)},$$

namely is the level of contribution rate necessary to cover the unfunded pension disbursements. Because it is assumed that $L^T(t) \geq F(t)$ for all values t in T, then $\alpha^{UN}(t)$ cannot take up negative values. If the pension system is fully funded, then it is $\alpha^{UN}(t) = 0$. Function $\alpha^{UN}(t)$ takes up values not greater than the pure PAYGO contribution rate, which is equal to $\frac{P(t)}{W(t)}$, as for each t in T it results in

$$\alpha^{UN}(t) = \frac{P(t)}{W(t)}\left(1 - D_c(t)\right).$$

It also is that

$$\alpha^{UN}(t)W(t) = \frac{L^{UN}(t)}{\gamma(t)\nu(t)} = P(t) - \frac{F(t)}{\gamma(t)\nu(t)},$$

which indicates that the "unfunded contributions" (at first side) are the contributions that pay for the unfunded pension expenditure.

Function $\alpha^F(t)$ is the level of funded contribution rate at time t, i.e.,

$$\alpha^F(t) = \alpha(t) - \alpha^{UN}(t) = \alpha(t) - \frac{\beta(t)}{\gamma(t)\nu(t)}. \qquad (4)$$

The level of funded contribution can take up negative values because it is dependent on the difference between the effective contribution rate, $\alpha(t)$, and the level of the unfunded contribution rate, $\alpha^{UN}(t)$. Relating to the "funded contributions", which are defined as $\alpha^F(t)W(t)$, the following relation is true

$$\alpha^F(t)W(t) = C(t) - \left(P(t) - \frac{F(t)}{\gamma(t)\nu(t)}\right),$$

namely the funded contributions are contributions that “remain”, in the algebraic sense, after the payment of the unfunded pension expenditure.

Fundamental sustainability conditions

To review the necessary and sufficient condition (NSC) of sustainability analysed in Angrisani 2008, a further definition is to be considered.

Function $intr(t)$ *is the intrinsic instantaneous rate of return at time t, i.e.,*

$$intr(t) = r(t) - \frac{1}{\gamma(t)\nu(t)}.$$

The following theorem is valid; see the proof in Angrisani 2008, pp. 72-74.

Theorem 1. NSC for pension system sustainability

Let a pension system have an initial non negative fund $F(t_*)$*, that is* $F(t_*) = F_* \geq 0$.

The pension system is sustainable in time interval $[t_*, t_f]$ *if and only if for each time* $t \in [t_*, t_f]$ *the whole of the funded contribution, paid until time* t *and discounted at time* t_* *by intrinsic instantaneous rate of return* $intr(t)$*, does not create a deficit greater than the initial available fund* $F(t_*)$*, i.e.,*

for each time $t \in [t_*, t_f]$ $\quad F(t) \geq 0$

if and only if

for each time $t \in [t_*, t_f]$ it results $\quad -\int_{t_*}^{t} e^{-\int_{t_*}^{\tau}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}\right)ds} \alpha^F(\tau) W(\tau) d\tau \leq F(t_*)$. (5)

We can explicate (5) of the NSC of sustainability in terms of unfunded pension expenditure. By definition (4), as $\alpha^F(t) = \alpha(t) - \alpha^{UN}(t)$, we have that (5) can be written for each time $t \in [t_*, t_f]$ as

$$\int_{t_*}^{t} e^{-\int_{t_*}^{\tau}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}\right)ds} \left(\alpha^{UN}(t) - \alpha(t)\right) W(\tau) d\tau \leq F(t_*)$$

or

$$\int_{t_*}^{t} e^{-\int_{t_*}^{\tau}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}\right)ds} \alpha^{UN}(t)W(\tau)\,d\tau \le F(t_*)+\int_{t_*}^{t} e^{-\int_{t_*}^{\tau}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}\right)ds} \alpha(t)W(\tau)\,d\tau.$$

Hence, we can say that if the pension system has an initial non negative fund, it is sustainable over the fixed time horizon if and only if, for each time t in T, the whole of the present values at time t_* of the unfunded pension expenditure, discounted at the intrinsic rate between t_* and t, is not greater than the initial fund plus the whole of the present values at time t_* of the effective contributions paid between t_* and t discounted at the intrinsic rate.

From the fundamental NSC of sustainability, the following sufficient condition of sustainability and the definition of the sustainability indicator are given, both in Angrisani 2008.

Proposition 1. Sufficient condition for pension system sustainability

It is assumed that $F(t_*)\ge 0$. *If the contribution rate* $\alpha(t)$ *is greater than or equal to the level of the unfunded contribution rate* $\alpha^{UN}(t)$ *for each time* $t\in\left[t_*,t_f\right]$, *then the pension system is sustainable in* $\left[t_*,t_f\right]$, *i.e.,*

if

$$\alpha(t)\ge\alpha^{UN}(t)=\frac{\beta(t)}{\gamma(t)\nu(t)} \text{ for each time } t\in\left[t_*,t_f\right] \qquad (6)$$

then

$F(t)\ge 0$ *for each time* $t\in\left[t_*,t_f\right]$.

Definition 3. *Function* $LSI(t)$ *is defined as the Logical Sustainability Indicator and for each time* $t\in\left[t_*,t_f\right]$ *is given by*

$$LSI(t)=\frac{C(t)\gamma(t)\nu(t)+F(t)}{L^T(t)}.$$

We note that, under the assumption of no negative initial fund, if for each t in T $LSI(t)\ge 1$, then the pension system is sustainable in T. In fact, if $LSI(t)\ge 1$ for each t in T, then

$\frac{C(t)\gamma(t)\nu(t)+F(t)}{L^T(t)} \geq 1$ for each t in T, hence $\alpha(t)W(t) \geq \frac{L^{UN}(t)}{\gamma(t)\nu(t)}$, and also $\alpha(t) \geq \frac{1}{\gamma(t)\nu(t)}\frac{L^{UN}(t)}{W(t)}$ for each t in T. Taking into account t the expression of $\beta(t)$ (see formula (3)), we have $\alpha(t) \geq \frac{\beta(t)}{\gamma(t)\nu(t)}$; it follows that condition (6) of Proposition 1 is satisfied, and, hence, the pension system is sustainable in T.

For its definition, indicator LSI is very similar to the balance ratio, which is "…a measure of the financial position of the Swedish pension system…", see SSIA 2009, p. 39. Indicator $\gamma(t)\nu(t)$ is the "natural" substitute of the Turnover Duration in the Swedish balance ratio. We note that, unlike the indicator $\gamma(t)\nu(t)$, the Turnover Duration is defined under the assumption of steady state conditions for the pension system, that is, under the assumption of "…stable population with stable income patterns…", Settergren and Mikula 2005. Under the hypothesis of a steady state, the two indicators, $\gamma(t)\nu(t)$ and Turnover Duration, coincide.

We have that *LSI* can also be expressed as

$$LSI(t) = D_c^{PAYGO}(t) + D_c(t) \quad \text{for each time } t \in \left[t_*, t_f\right]$$

which highlights the integrated role of the two different financing methods, the PAYGO one and the funded one. In fact, if $D_c^{PAYGO}(t)$ is greater than or equal to 1 – that is, if the current contributions are able to pay pension expenditures – then the pension system is obviously sustainable in T. Otherwise, if $D_c^{PAYGO}(t)$ falls below 1, then the deficiency of the system is compensated by means of the funded component.

3. THE NECESSARY CONDITION OF SUSTAINABILITY

In this section we give the condition which determines the "minimum level" of the constant contribution rate only necessary, but not sufficient, for the sustainability of the pension system which is provided with a funded component and is in the stabilisation phase. In this condition we assume an efficient rule on the rate of return on the pension liability, which allows to stabilise the indicator $\beta(t)$.

First, we review this fundamental rule, see Angrisani 2008.

Proposition 2. (Stabilisation of indicator $\beta(t)$)

It is assumed that $0 \le F(t_*) < L^T(t_*)$. *For each* $t \in [t_*, t_f]$, *it results* $\beta(t) = \beta(t_*)$ *if and only if*

$$r_L(t) = D_c(t) r(t) + (1 - D_c(t)) \frac{\dot{W}(t)}{W(t)}. \qquad (7)$$

The rate of return on the pension liability in (7) is given according to the "*rule of the weighted average"* as this equals the convex combination of the assets interest rate and wages growth rate with weights proportional to the funded and unfunded components of the total pension liability. It has to be emphasised that the rate of return earned explicitly by the pension credit of retirees has to consider the instantaneous rate of the implicit demographic return.

In advance, we note that function $\gamma(t)\nu(t)$, which will be used in the following theorem, is dependent on the underlying demographic structure and the complex of the variables of the pension system, including the contribution rate.

Furthermore, we pose $W(t) = W(t_*) e^{\int_{t_*}^{t} \sigma(s)\, ds}$ for each *t* in *T,* where $\sigma(s)$ is the instantaneous growth rate in wages.

Theorem 2. Necessary condition of pension system sustainability under the assumption of a constant contribution rate and in the stabilisation phase

Let T be the time interval $T = [t_*, +\infty)$. *Let the pension system be of defined contribution type with an initial fund* $F(t_*)$ *such that* $F(t_*) = F_* \ge 0$.

Let contribution rate $\alpha(t)$ *be constant over time, that is* $\alpha(t) = \overline{\alpha}$ *for each time t in T.*

We assume that for each t in T the "rule of the weighted average" on the rate of return on the pension liability is used, see (7).

Let $\gamma(t)\nu(t)$ *be the divisor of the pension liability in the pension expenditure such that*

$$\lim_{t \to +\infty} \gamma(t)\nu(t) = \overline{\lambda}, \qquad (8)$$

where $\overline{\lambda}$ is independent on the contribution rate.

Furthermore, it is assumed that for each t in domain T the below integral exists and it is

$$\lim_{t\to+\infty}\int_{t_*}^{t} e^{\int_{t_*}^{\tau}\left(\sigma(s)-\left(r(s)-\frac{1}{\gamma(s)\nu(s)}\right)\right)ds} d\tau = +\infty . \qquad (9)$$

Then the necessary condition for the pension system sustainability in T is that the contribution rate satisfies condition

$$\overline{\alpha} \geq \frac{\beta_*}{\overline{\lambda}} \text{ where } \beta_* = \beta(t_*). \qquad (10)$$

Proof. We assume that the pension system is sustainable with constant contribution rate $\alpha(t)=\overline{\alpha}$, namely

$$F(t) \geq 0 \text{ for each } t \text{ in } T. \qquad (11)$$

According to the necessary and sufficient condition of sustainability – see Theorem 1 – one has that

$$-\int_{t_*}^{t} e^{-\int_{t_*}^{\tau}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}-\sigma(s)\right)ds} \alpha^F(\tau)\,d\tau \leq \frac{F(t_*)}{W(t_*)} \text{ for each } t \text{ in } T.$$

By definition, see (4), one has $\alpha^F(\tau)=\alpha(\tau)-\alpha^{UN}(\tau)$ and, having assumed that the contribution rate is constant over time, one has $\alpha^F(\tau)=\overline{\alpha}-\alpha^{UN}(\tau)$. Hence, it is

$$\frac{F(t_*)}{W(t_*)} \geq \int_{t_*}^{t} e^{-\int_{t_*}^{\tau}\left(r(s)-\frac{1}{\gamma(s)\nu(s)}-\sigma(s)\right)ds} \left(\alpha^{UN}(\tau)-\overline{\alpha}\right)d\tau \text{ for each } t \text{ in } T. \qquad (12)$$

By hypothesis, the rate of return on the pension liability is chosen according to the "rule of the weighted average", hence, from Proposition 2, one has that, for each *t* in *T,* $\beta(t)=\beta(t_*)=\beta_*$.

Consequently, for each *t* in *T* one has that $\alpha^{UN}(t)=\dfrac{\beta_*}{\gamma(t)\nu(t)}$.

Therefore, taking into account (8), it results in $\lim_{t\to+\infty}\alpha^{UN}(t)=\dfrac{\beta_*}{\overline{\lambda}}$.

If, ad absurdum, it is $\overline{\alpha}<\dfrac{\beta_*}{\overline{\lambda}}$, then it is

$$\lim_{t\to+\infty}\left(\alpha^{UN}(t)-\overline{\alpha}\right)=\lim_{t\to+\infty}\alpha^{UN}(t)-\overline{\alpha}=\frac{\beta_*}{\overline{\lambda}}-\overline{\alpha}=l ,$$

where l is a real number greater than zero.

Then, it is definitively, i.e. there is a $\bar{t}$ in T such that

$$\text{for each } t > \bar{t} \text{ it is } \alpha^{UN}(t) - \bar{\alpha} > \frac{l}{2}. \qquad (13)$$

Therefore, by (12) it follows that for each t in T such that $t > \bar{t}$ it is

$$\frac{F(t_*)}{W(t_*)} \geq \int_{t_*}^{\bar{t}} e^{-\int_{t_*}^{\tau}\left(r(s) - \frac{1}{\gamma(s)\nu(s)} - \sigma(s)\right)ds} \left(\alpha^{UN}(\tau) - \bar{\alpha}\right) d\tau + \int_{\bar{t}}^{t} e^{-\int_{t_*}^{\tau}\left(r(s) - \frac{1}{\gamma(s)\nu(s)} - \sigma(s)\right)ds} \left(\alpha^{UN}(\tau) - \bar{\alpha}\right) d\tau$$

and hence, by (13), for each t in T such that $t > \bar{t}$ it is

$$\frac{F(t_*)}{W(t_*)} > \int_{t_*}^{\bar{t}} e^{-\int_{t_*}^{\tau}\left(r(s) - \frac{1}{\gamma(s)\nu(s)} - \sigma(s)\right)ds} \left(\alpha^{UN}(\tau) - \bar{\alpha}\right) d\tau + \frac{l}{2} \int_{\bar{t}}^{t} e^{-\int_{t_*}^{\tau}\left(r(s) - \frac{1}{\gamma(s)\nu(s)} - \sigma(s)\right)ds} d\tau$$

which is impossible by hypothesis (9). Therefore, assumption $\bar{\alpha} < \frac{\beta_*}{\bar{\lambda}}$ must be false and, hence, $\bar{\alpha} \geq \frac{\beta_*}{\bar{\lambda}}$.

End of proof.

Numerical exemplification

We consider a defined contribution pension system which has a funded component at the initial time t_* of observation. We assume that the contribution rate is constant over time.

We set the values at time t_* of the considered variables of the pension system as in Table 1. All the values are expressed according to a same money unit. Without going into the merits of a specific sustainability analysis of the Swedish pension system, hence exclusively as numerical exemplification, for our illustration we take into consideration the initial data similar to those of the Swedish pension system (see the Orange Report, SSIA 2009). We observe that the data published in the Orange Report are frequently smoothed respect to the effective data of the year.

Table 1: Initial values of the variables of the pension system

Pension system variables	Initial values
Total pension liability	7500
Fund	825
Pension contributions	200
Pension expenditure	217

On the base of the previous data, we evaluate the initial values of the indicators proposed in our model, see following Table 2.

Table 2: Initial values of the indicators of the pension system

Pension system indicators	Initial values
Divisor of the total pension liability in the pension expenditure	34.56
Level of the unfunded pension liability respect to wages	5.34
Level of the unfunded contribution rate	15.45%
Degree of funding of the pension liability	11.00%
Degree of PAYGO covering of the pension expenditure by contributions	92.17%
Logical sustainability indicator	1.0317

In our example we want to apply the necessary condition of sustainability given by Theorem 2.

We use the "rule of the weighted average" for the rate of return on the pension liability to obtain the stabilisation of $\beta(t)$ at its initial value, namely 5.34.

We want that, in our example, assumption (9) of Theorem 2 is valid, that is

$$\lim_{t \to +\infty} \int_{t_*}^{t} e^{\int_{t_*}^{\tau} \left(\sigma(s) - \left(r(s) - \frac{1}{\gamma(s)\nu(s)} \right) \right) ds} d\tau = +\infty .$$

We have that if the difference between the instantaneous rates, $r(s)$ and $\sigma(s)$, is less than or equal to a value lower to limit $\frac{1}{\overline{\lambda}}$ (as a matter of fact, it is sufficient that the difference between the integral average values of such rates is less than or equal to a value lower to previous limit $\frac{1}{\overline{\lambda}}$), then condition (9) is verified. In particular, (9) is valid if we assume, in our example, the following choices.

We assume that limit value $\overline{\lambda}$ is the value of the smoothed Turnover Duration of the Swedish pension system in 2009, which is 31.76198, see the Orange Report, SSIA 2009.

We assume that the two rates, the instantaneous rate of return on assets and the instantaneous growth rate in wages (assumed equal to the rate of growth in average income, under hypothesis of stable number of active population), are chosen equivalent to the corresponding real annual rates given in the Pessimistic Scenario in the Orange Report 2009, p. 24. Therefore, both $r(s)$ and $\sigma(s)$ are assumed to be constant and equal to 0.995%. In this case, difference $r(s)-\sigma(s)$ is zero and, hence, strictly less than $\frac{1}{\overline{\lambda}} = 3.1484\%$.

With this choice of the instantaneous rates all the hypotheses of the Necessary Condition are satisfied in our example, including the invariance of $\overline{\lambda}$ respect to the contribution rate (as we will show in a next work). Hence, the level of the constant contribution rate, which is only necessary for the sustainability, must not be less than value $\frac{\beta_*}{\overline{\lambda}} = 16.81\%$.

4. CONCLUSION

The present paper gives a necessary condition on the sustainability for defined contribution pension systems provided with a funded component under the assumption of a constant contribution rate and in the stabilisation phase. The methodology of financial management, produced in two previous works of one of the two authors, is applied. In particular, the rate of return on the pension liability is naturally assumed according to the "rule of weighted average", see Angrisani 2006. Furthermore, the necessary and sufficient condition of sustainability, produced in Angrisani 2008, is used as only necessary condition. On these elements, we have proved the condition relative to the "minimum level" of the contribution rate, assumed to be constant, which is only necessary so that a pension system is sustainable over time in the stabilisation phase. A numerical example of the necessary condition application is also given

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